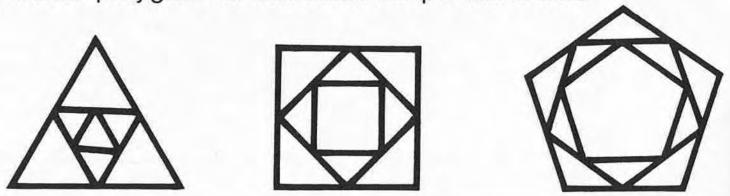
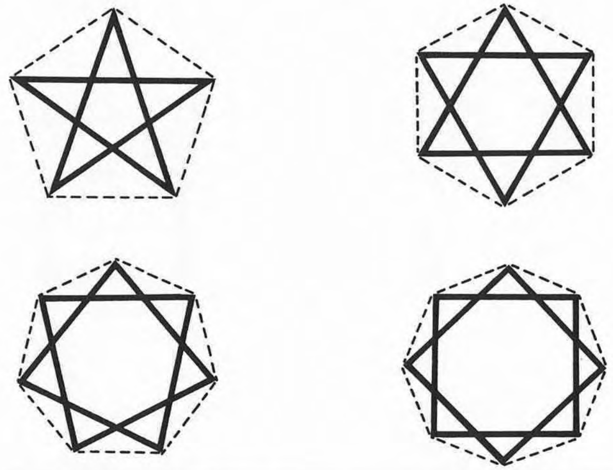
If the edges of a regular polygon are bisected, or cut in half, and the points of bisection joined between adjacent edges, smaller and smaller polygons of the same shape will result.



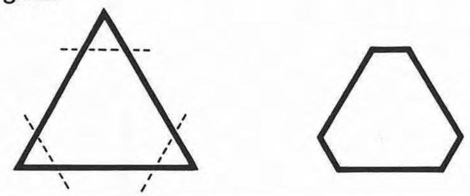
Star Polygons

Star polygons can be formed from polygons of more than five sides either by connecting alternate vertices of the polygon, or by extending the sides of the polygon until they intersect.

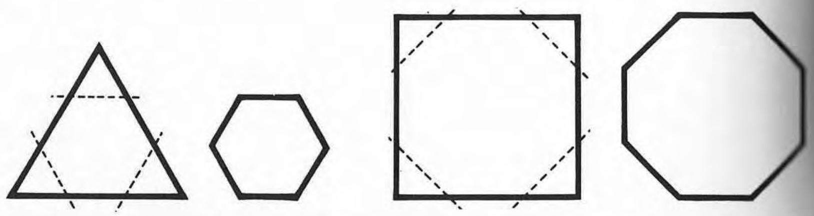


Truncating Polygons

A truncated polygon is one whose corners have been cut off. Truncation forms a new polygon with more sides than the original polygon.



A regular polygon can be truncated to form a new regular polygon.

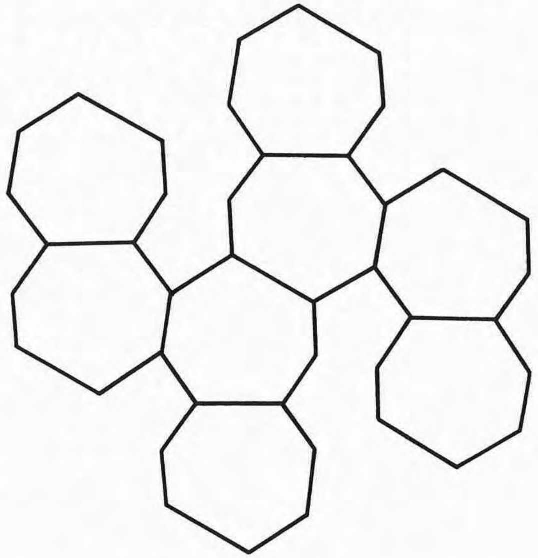


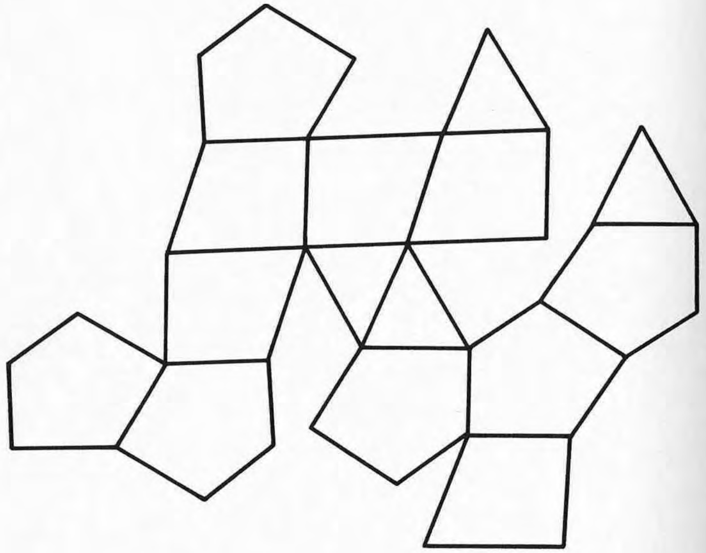
2

Tessellations

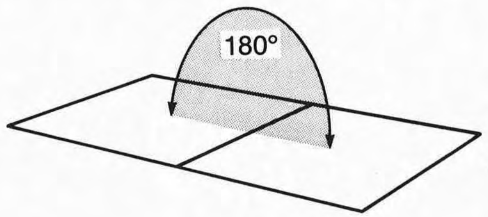
Combining Polygons

In the plane, polygons can be joined together along matching edges.



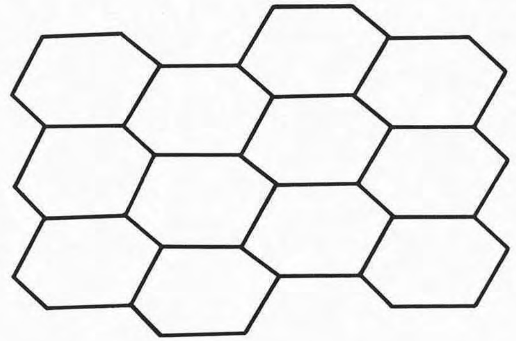


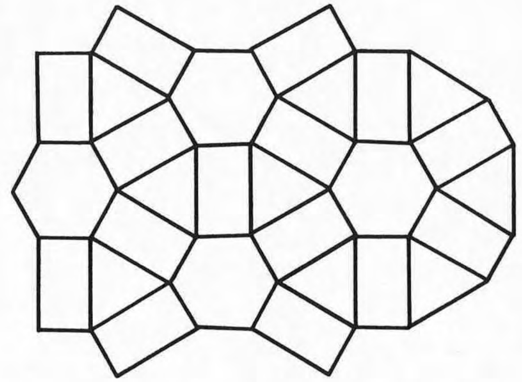
In the plane, the angle between joined polygons is 180°.

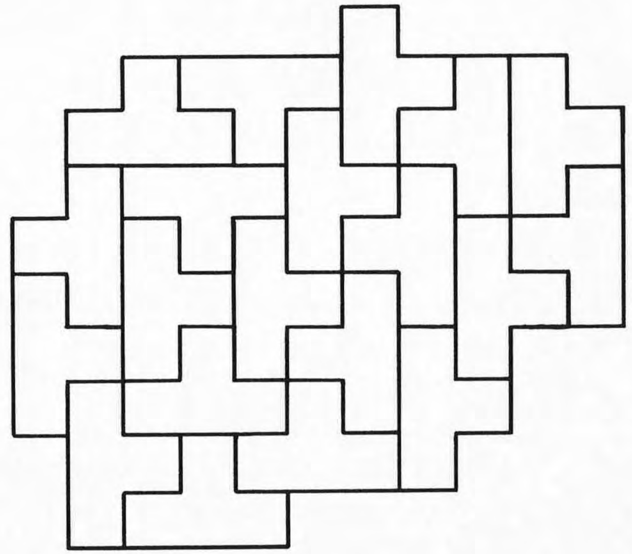


Tile Tessellation

Polygons tile the plane if they fit together without overlaps or gaps. The resulting pattern is a tessellation or tiling.

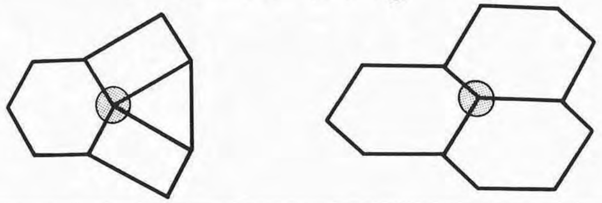






Vertex

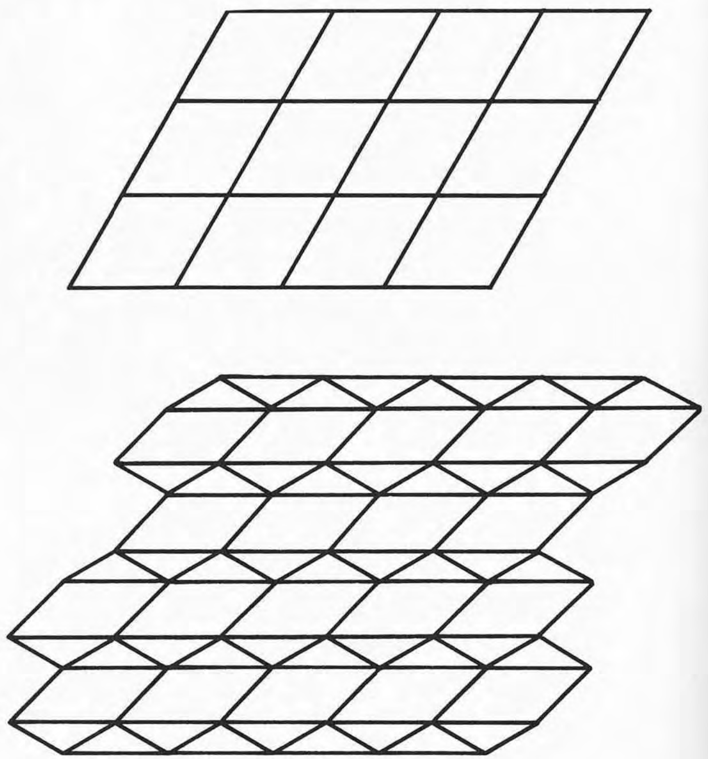
In a tessellation, a vertex is formed when three or more sides of the polygons meet at a single point. Three sides meeting at a vertex is the minimum condition for tiling.



In a tessellation, the sum of the angles around each vertex is always equal to 360°.

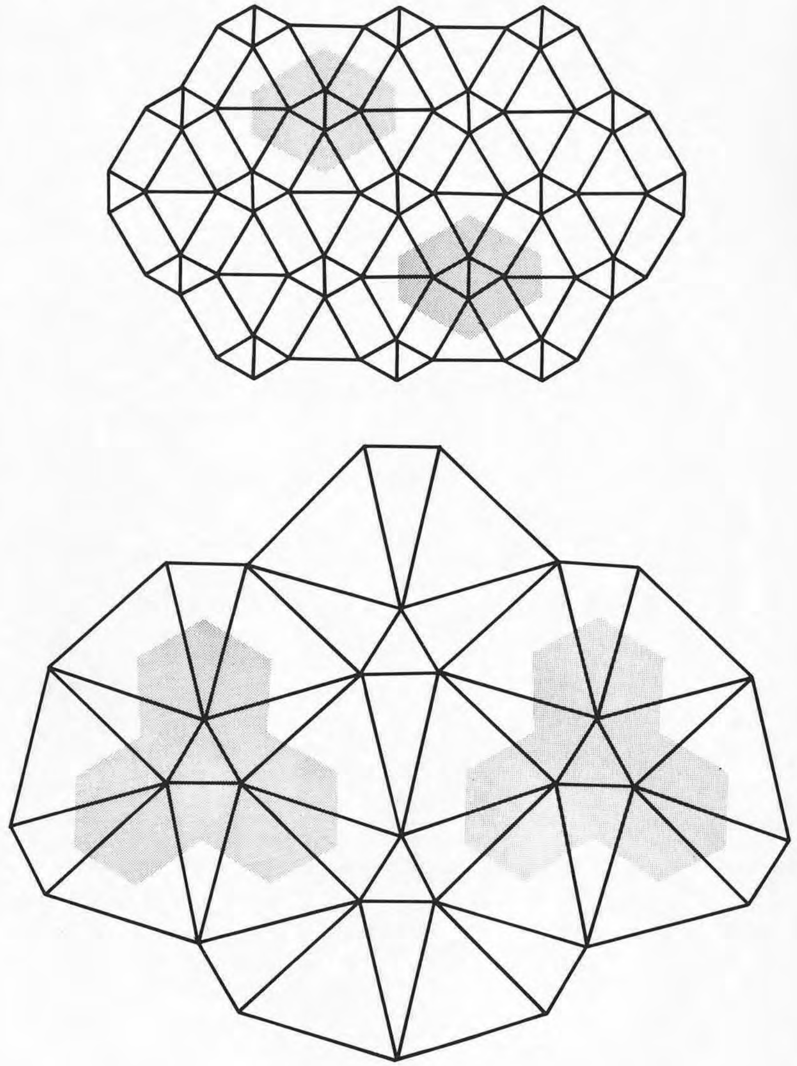
Uniform Tiling

In a uniform tiling, all vertices are congruent. That is, the arrangement of polygons around every vertex is the same.



Periodic Tiling

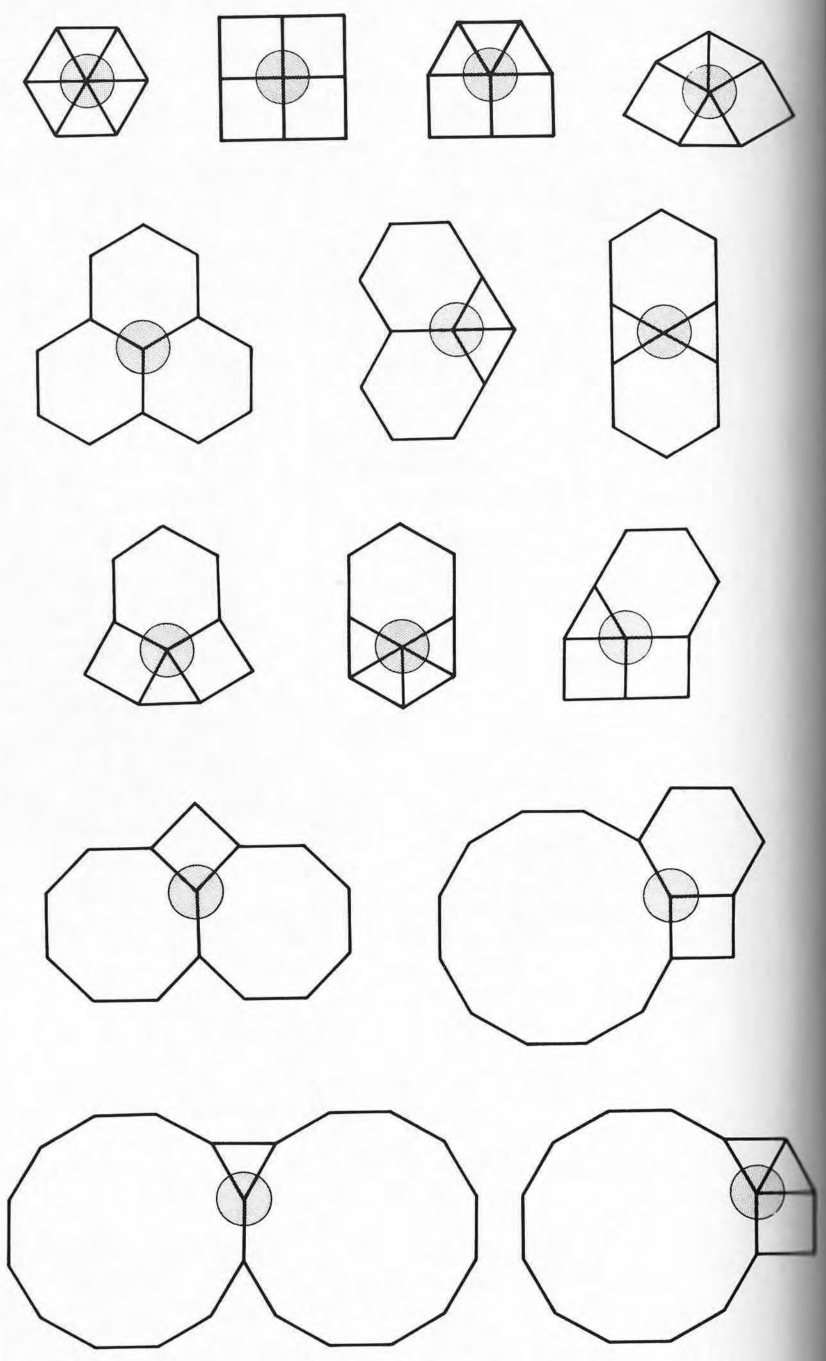
In a periodic tiling, any region can be shifted to a new position where it again fits exactly. This region may be moved in any direction as long as it is not rotated.



Tiling with Regular Polygons

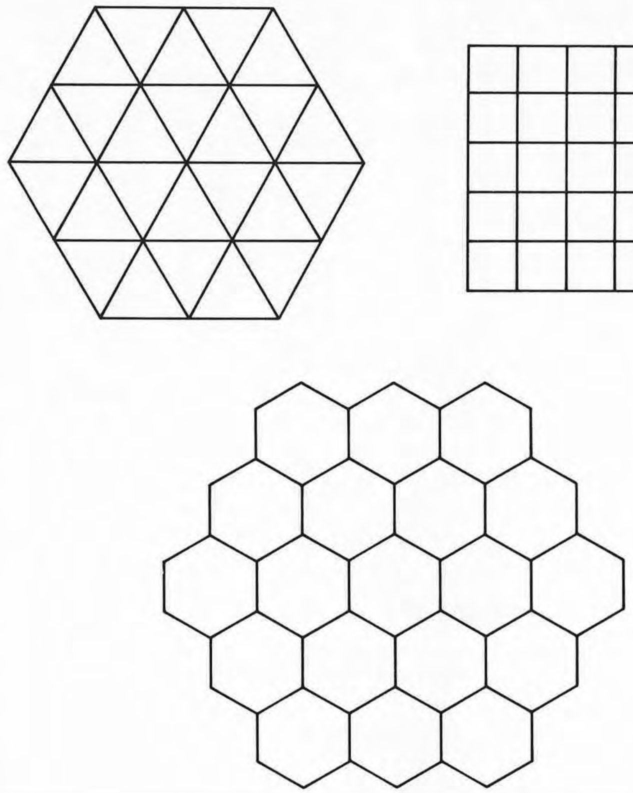
Typical Vertices

Of the regular polygons, only triangles, squares, hexagons, octagons, and dodecagons can tile in various combinations around a common vertex. There are only 14 such combinations or typical vertices.

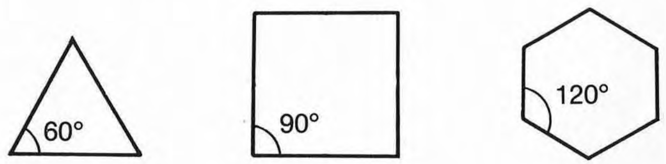


Regular Tessellations

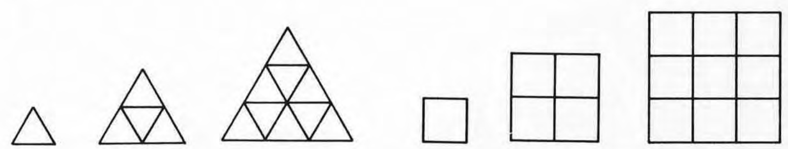
A regular tessellation is periodic and uniform and consists of congruent regular polygons.



There are only three regular tessellations because there are only three regular polygons whose face angles divide evenly into 360°.

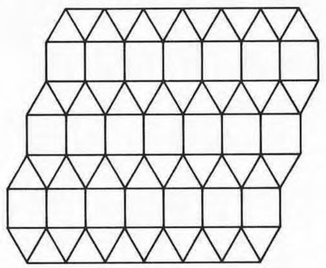


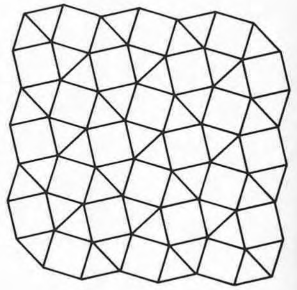
Equilateral triangles and squares can be repeated to form larger versions of themselves.

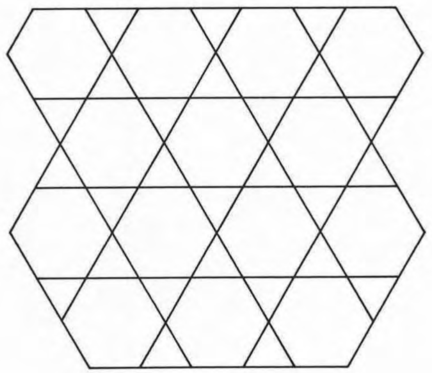


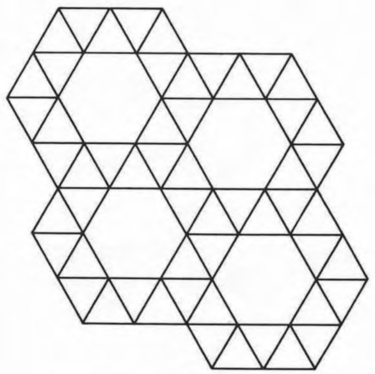
Semiregular Tessellations

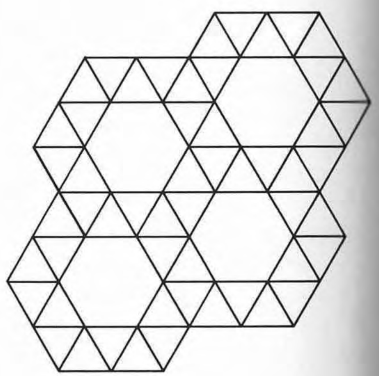
A semiregular tessellation is periodic and uniform and consists of more than one kind of regular polygon. There are eight semiregular tessellations. One of them is an enantiomorph.

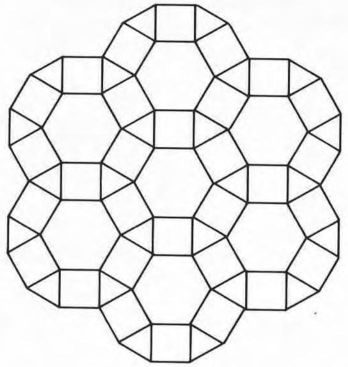


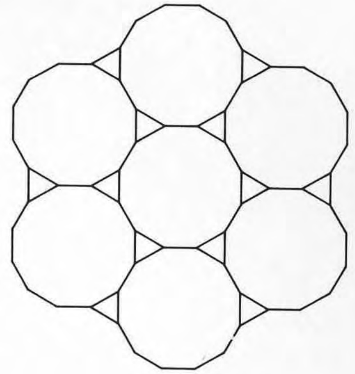


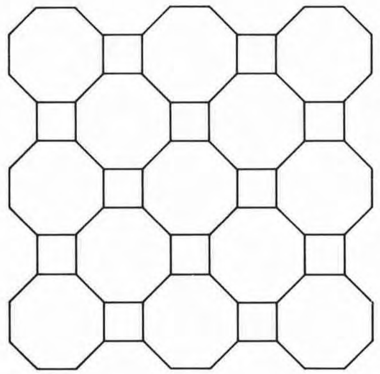


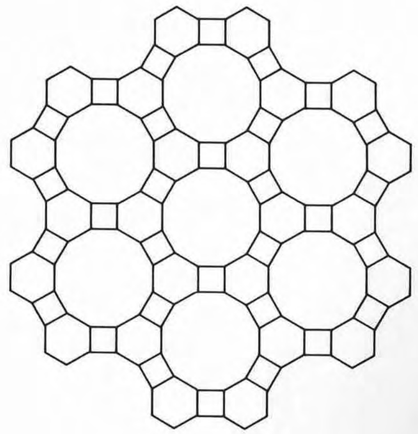






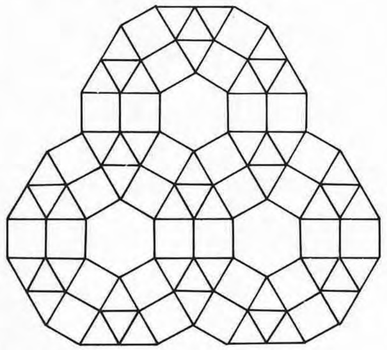


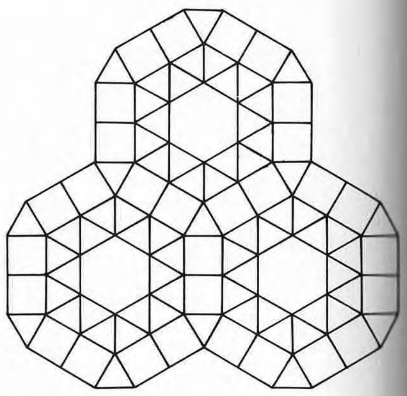


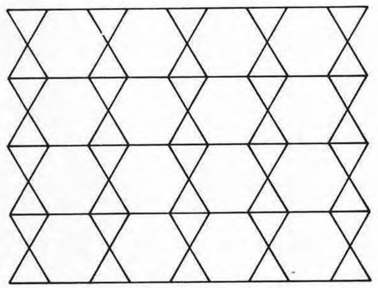


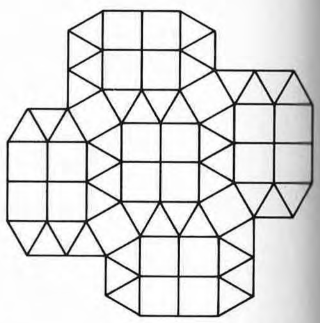
Nonuniform Periodic Tessellations with Regular Polygons

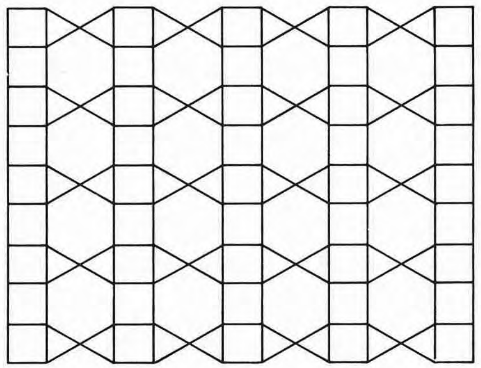
There is an infinite number of nonuniform periodic tessellations with regular polygons. However, because 360° is required around each vertex, we are still restricted to the 14 typical vertices for regular polygons.

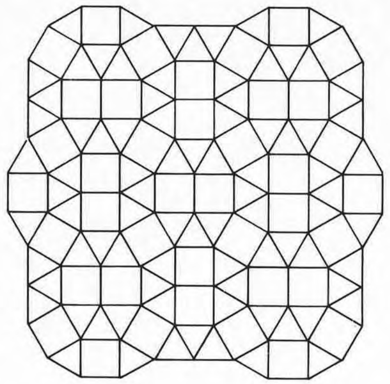


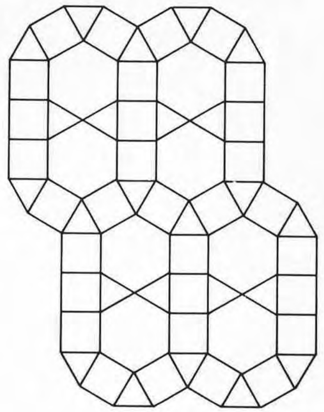


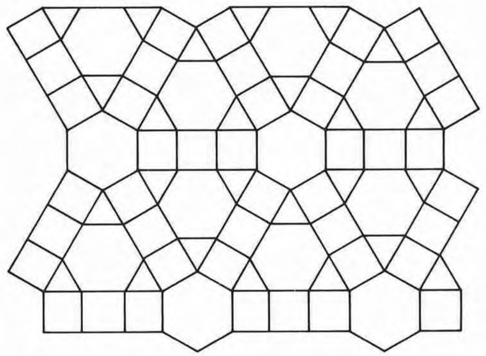


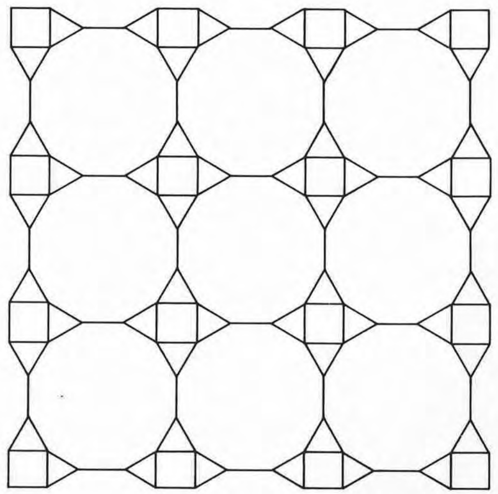






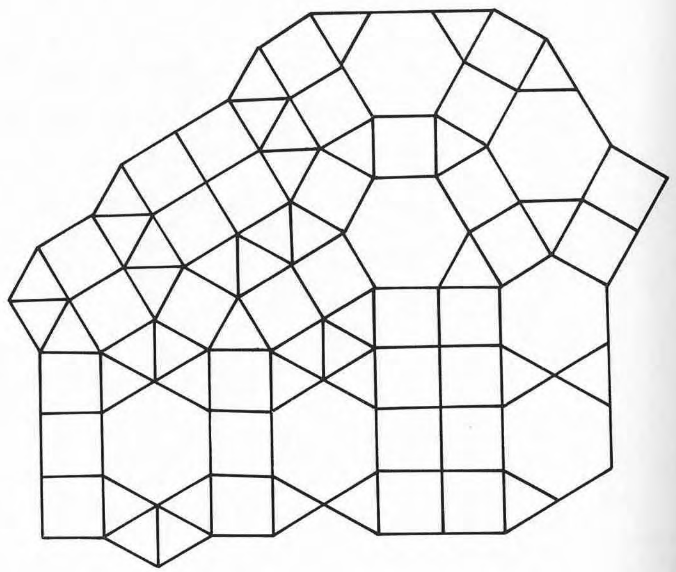


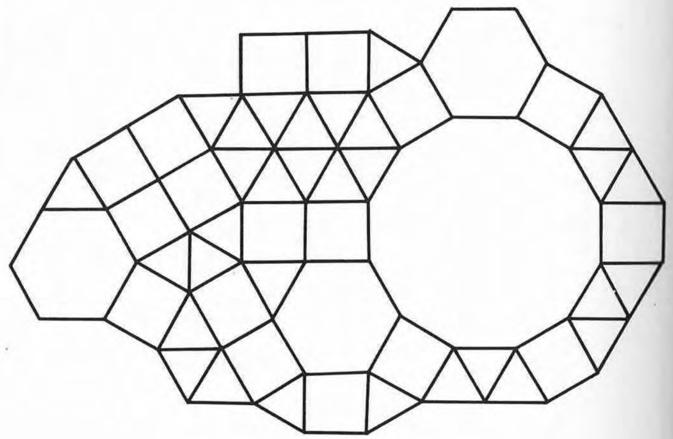




Nonuniform Nonperiodic Tessellations with Regular Polygons

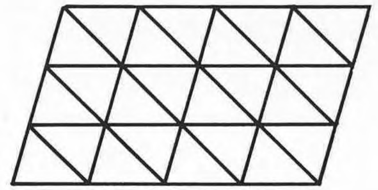
There is an infinite number of nonuniform nonperiodic tessellations with regular polygons. Again, we are restricted to the 14 typical vertices.



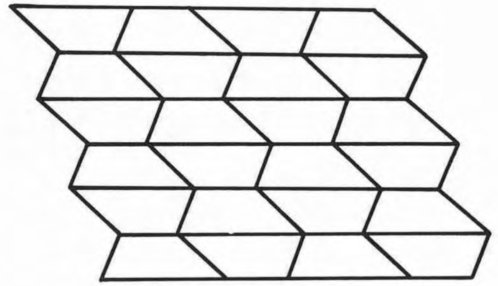


Tiling with Nonregular Polygons

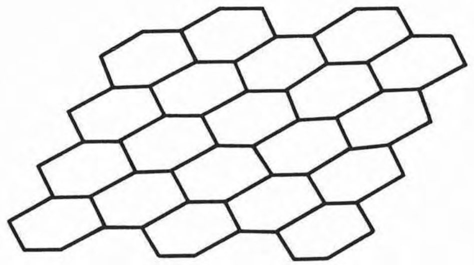
Any triangle tiles the plane.



Any quadrilateral tiles the plane.

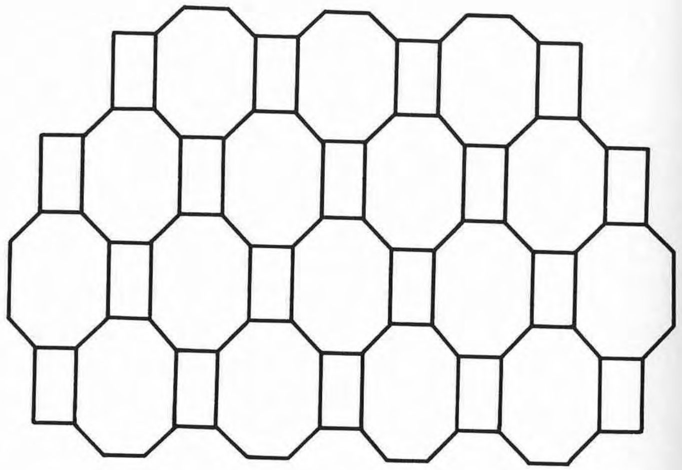


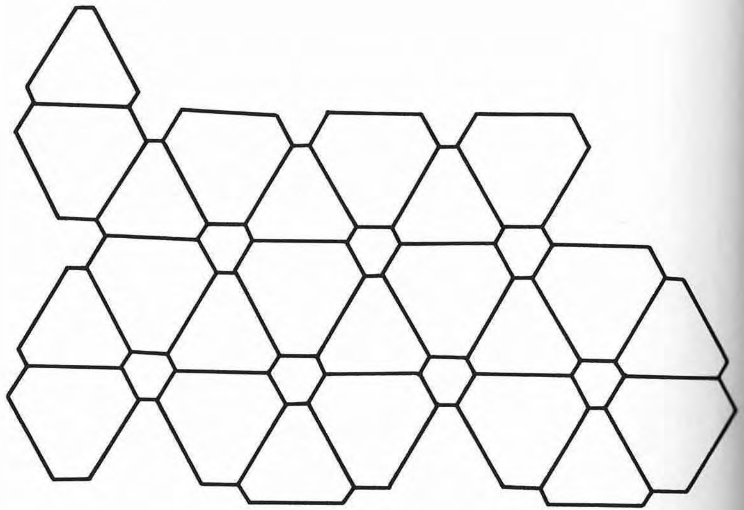
Any hexagon with three sets of equal parallel sides tiles the plane.



Uniform Periodic Tessellations with Nonregular Polygons

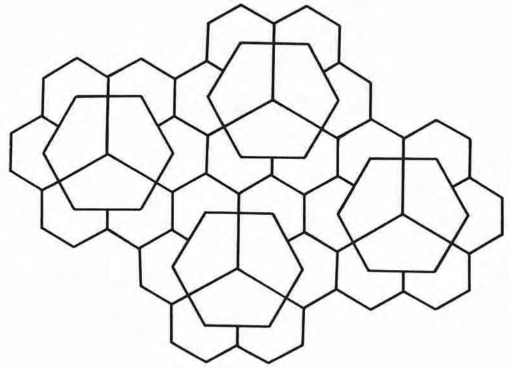
There is an infinite number of uniform periodic tessellations with nonregular polygons.

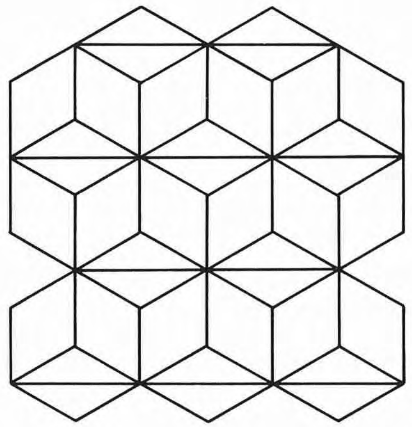


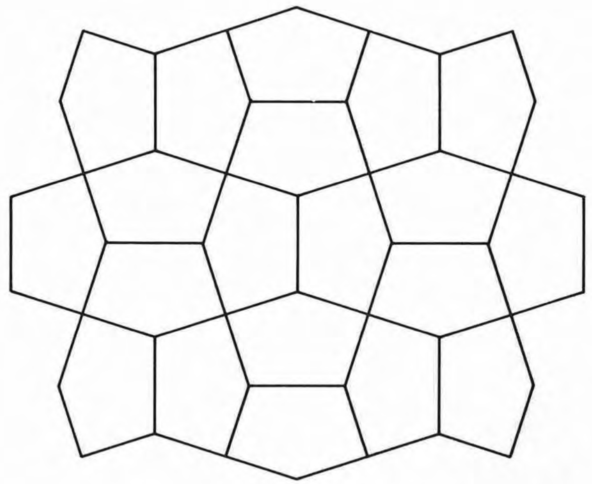


Nonuniform Periodic Tessellations with Nonregular Polygons

There is an infinite number of nonuniform periodic tessellations with nonregular polygons.

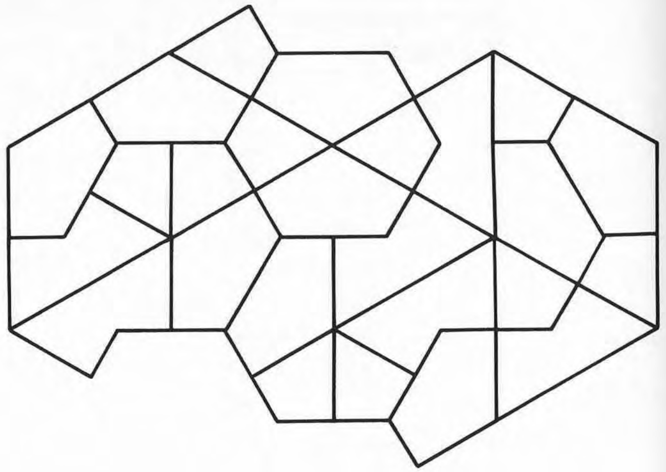


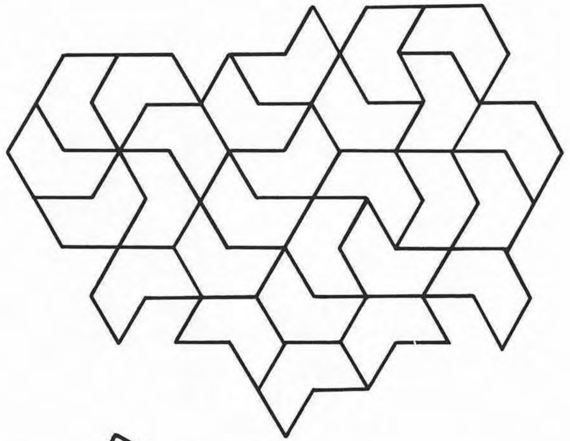


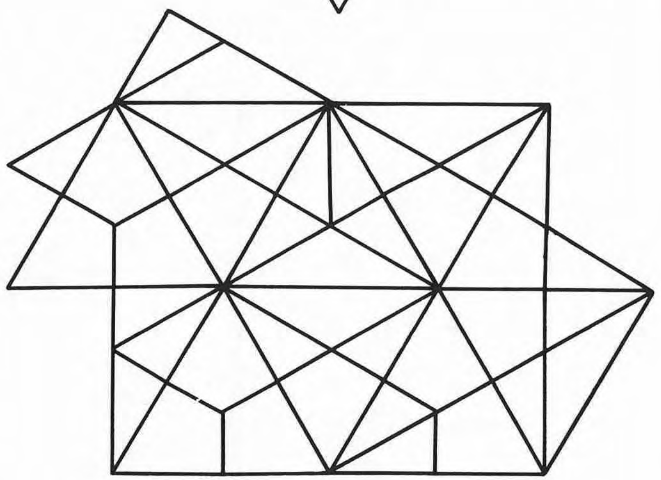


Nonuniform Nonperiodic Tessellations with Nonregular Polygons

There is an infinite number of nonuniform nonperiodic tessellations with nonregular polygons.

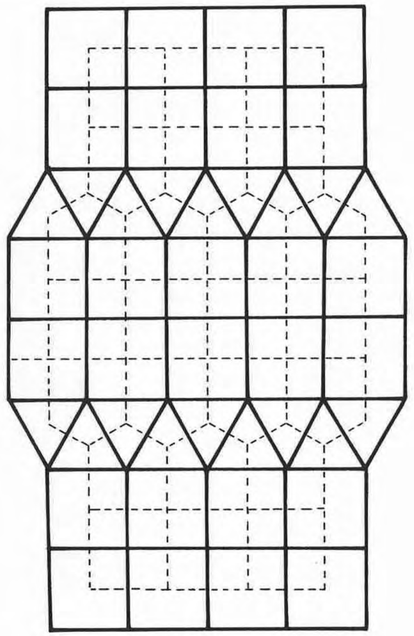




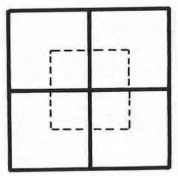


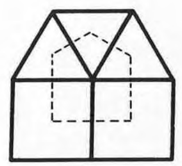
Dual Tessellation

A dual tessellation of a tiling is formed by joining the center of each polygon through its sides to the centers of all neighboring polygons. A dual tessellation has as many polygons as the original tessellation has vertices, and as many vertices as the original has polygons. The number of sides remains the same.

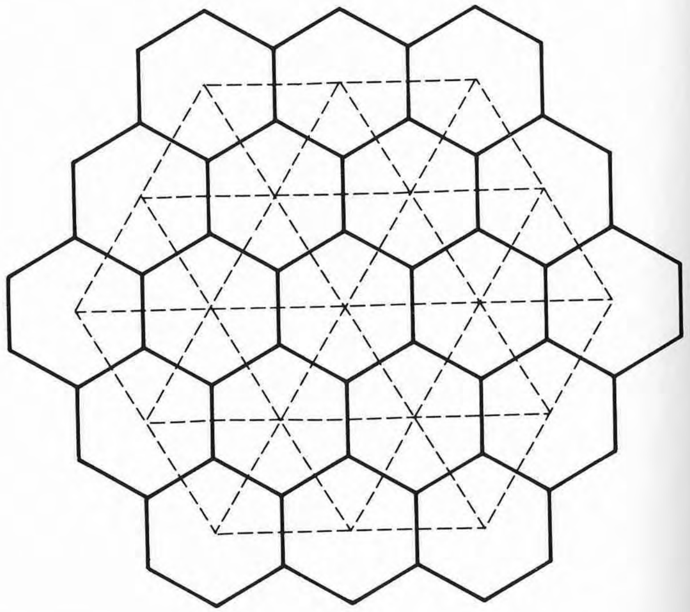


A polygon formed by a dual tessellation has the same number of sides as there are edges meeting in its center.



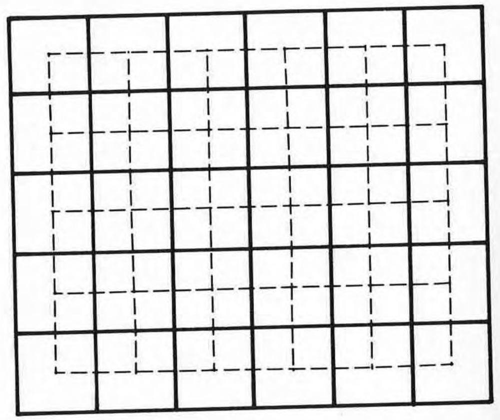


The two regular tessellations of triangles and hexagons are dual to each other.



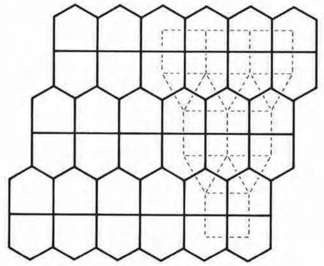
Self-Dual

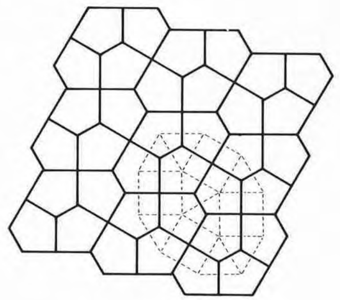
A self-dual forms the same polygons as comprise the original tessellation. The regular tessellation of squares is a self-dual.

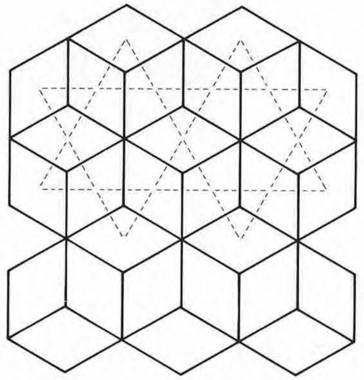


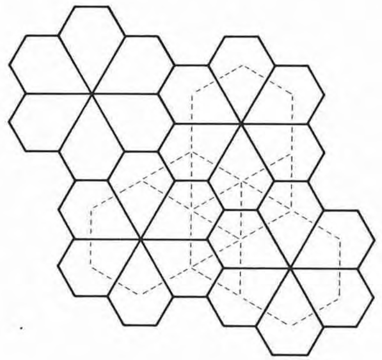
Dual Tilings of the Semiregular Tessellations

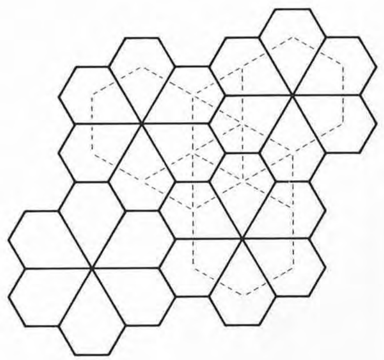
The dual tilings of the semiregular tessellations form polygons that are all nonregular. Because the semiregular tessellations are uniform, the polygons formed by the dual tiling are congruent or enantiomorphic. The dual tilings are periodic but not uniform.

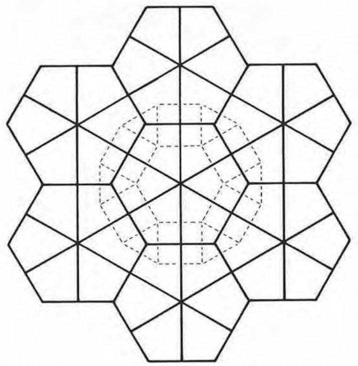


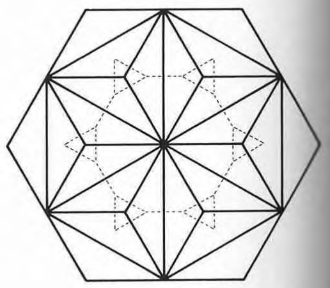


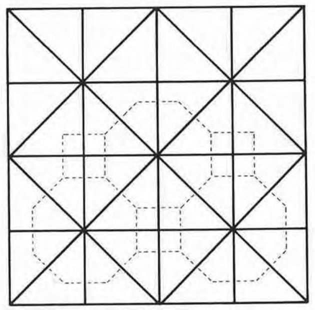


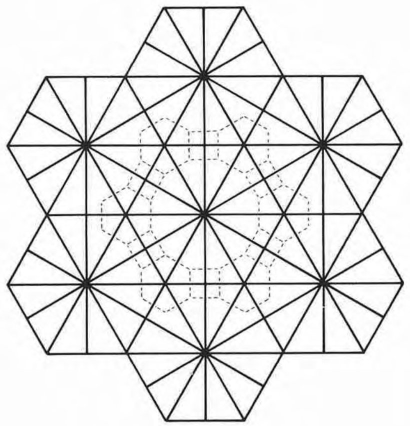




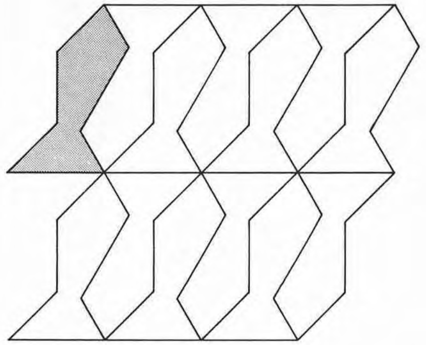


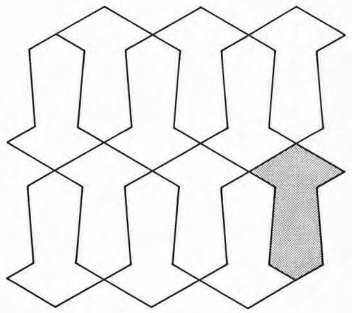


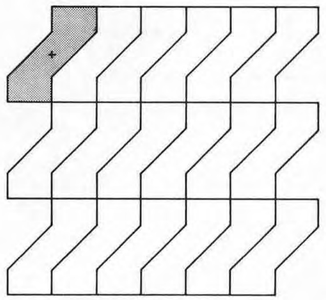


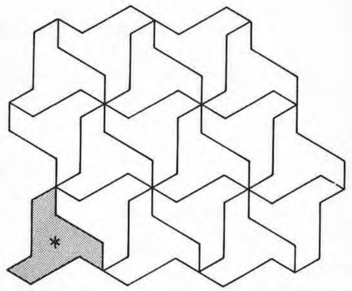


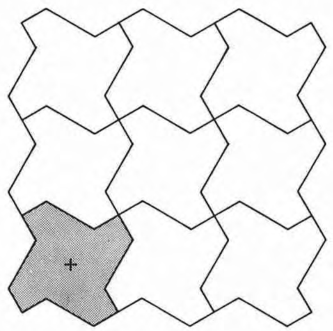
In a tiling with congruent polygons, only polygons with the following symmetries may be used: no symmetry, mirror symmetry, 2-fold, 3-fold, 4-fold, and 6-fold. Polygons with other symmetries, such as 5-fold, will not tile the plane.

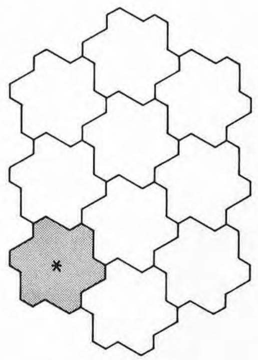








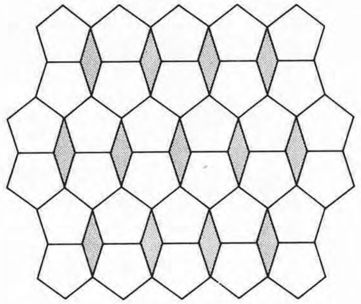


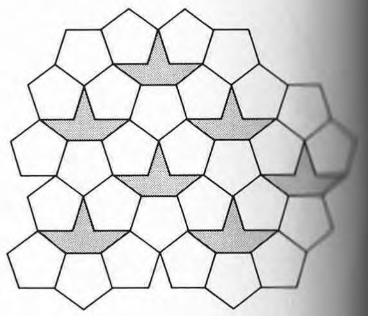


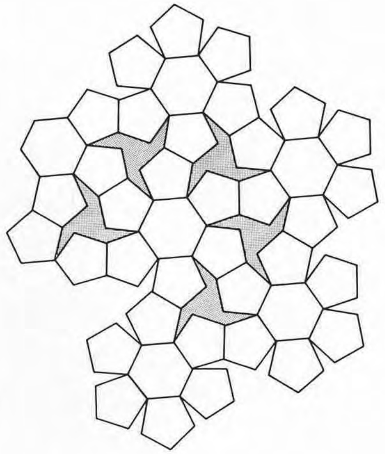
Open Patterns with Regular Polygons

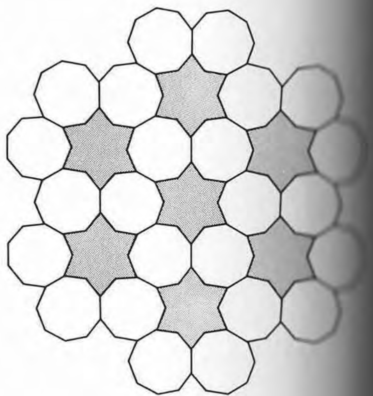
Periodic Patterns

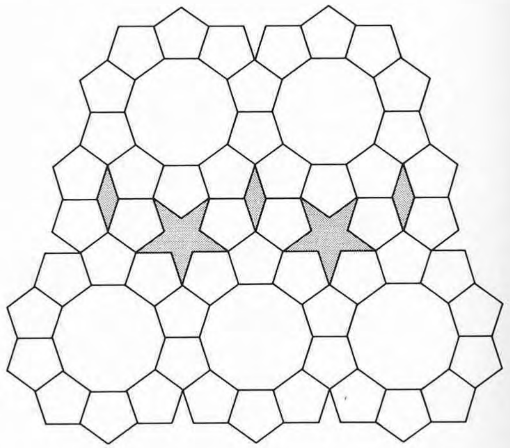
The plane can be subdivided periodically without the requirement of filling the whole plane. In such cases, regular polygons of 5, 9, 10, and 20 sides may be used. Use of these polygons results in open spaces of nonregular polygons.





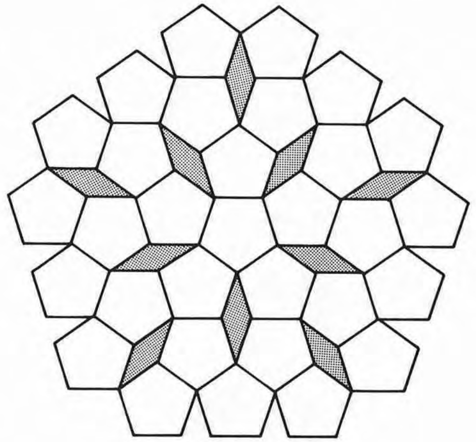


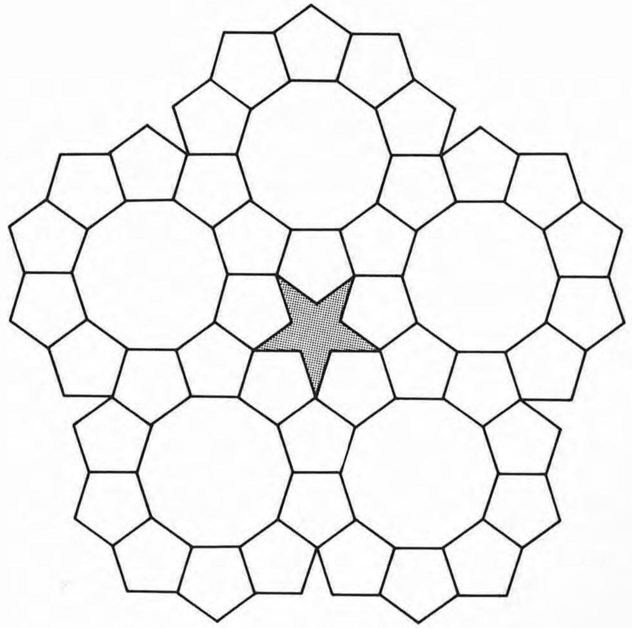




Concentric Patterns

Though the regular pentagon does not tile the plane, it can be used for open patterns that form concentrically around a central figure.



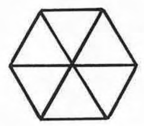


Euler's Theorem

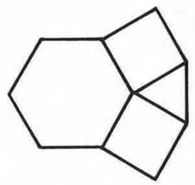
Euler's theorem for tilings demonstrates that there is always a consistent relationship among the components of a tessellation: the number of polygons + the number of vertices = the number of edges + 1.

$$P + V = E + 1$$

$$6 + 7 = 12 + 1$$
$$13 = 13$$



$$4 + 10 = 13 + 1$$
$$14 = 14$$



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Polyhedra